Comment on "Breakdown of the tensor component in the Skyrme energy density functional"†

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In a recent paper published in Phys. Rev. C[1], hereafter referred to as Dong and Shang, the authors claim in the abstract that "the Skyrme original tensor interaction is invalid." One should note that, although the original idea by Skyrme dates back to the 1950s and the Skyrme tensor force has been written out in Ref. 2), there have been many studies on how to implement it and fit its parameters since then, even in recent years. In 2014, some of us published a review paper on the tensor force within mean-field and density functional theory (DFT) approaches to the nuclear structure.[3] In that review, we quoted ≈30–40 papers where the Skyrme tensor is implemented and studied, authored by different colleagues. The conclusion of that review paper is that evidences for a strong neutron-proton tensor force exist, even in mean-field and DFT studies. Therefore, if the tensor force proposed by Skyrme were "invalid," this would impact on a considerable number of published works and conclusions drawn so far. Accordingly, it would be appropriate to have a strong argument regarding this "invalidity."

The argument against the Skyrme tensor force in Ref. 1) was given as: "The Skyrme original tensor force was introduced in an unreasonable way, because the tensor-force operator S_{12} in momentum space but with an r-dependent strength, *i.e.*, $f_T(r)S_{12}(\mathbf{k})$, is applied as a starting point." We note that $S_{12}(\mathbf{k})$ can be expressed as, $S_{12}(\mathbf{k}) = (\sigma_1 \cdot \mathbf{k})(\sigma_2 \cdot \mathbf{k}) - k^2 \frac{\sigma_1 \cdot \sigma_2}{3}$. We do not see any logic behind the sentence by Dong and Shang. In physics, nothing forbids an interaction to be position-dependent and momentum-dependent at the same time. Bethe[4] was one of the first to advocate that this ought to be the case if one wishes to introduce an effective potential for finite nuclei. Skyrme echoed this at the beginning of his first paper on this topic, and it is useful at this stage to quote literally the sentence from Ref. 5): "in the case of a finite system the effective potential must depend upon both momenta and coordinate."

Several authors have proposed Skyrme-type forces with terms that have both momentum- and density-dependence. If the Skyrme tensor force is unreasonable because of the reason advocated by Dong and Shang, the whole Skyrme force will be unreasonable. If this were the case, this would discredit not only the results of a few hundred papers in which Skyrme tensor terms are introduced but also some more $\approx 10^3$ papers in which Skyrme forces are used.

In the paper by Dong and Shang, it is also claimed that the Skrme-type tensor interaction is introduced in an unreasonable way since the Fourier transform of such tensor interaction is difficult or even impossible. We disagree on this claim based on the procedure that we discuss explicitly as follows. We show that if we start from a general central, spin-orbit or tensor interaction with a radial dependence such that the range is very short, the Fourier transform produces the Skyrme force in momentum space. Let us write the Fourier transform of an interaction $V(r)$ as

$$V(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}} V(r) d\mathbf{r}, \tag{1}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer ($\mathbf{k}$ and $\mathbf{k}'$ are the initial and final relative momenta, respectively). The tensor interaction in the coordinate space is expressed as

$$V_T(r) = f_T(r) S_{12}(\mathbf{r}), \tag{2}$$

where $S_{12}(\mathbf{r}) = (\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r}) - r^2 \frac{\sigma_1 \cdot \sigma_2}{3} = \sqrt{\frac{8\pi}{3}}[(\sigma_1 \cdot \sigma_2)^{(2)} \times r^2 Y_2(\hat{r})]^{(0)}$ with the spherical harmonics $Y_{2\mu}$. The Fourier transform can be evaluated as

$$V_T(\mathbf{q}) \propto \int r^4 dr f_T(r) j_2(qr) [(\sigma_1 \cdot \sigma_2)^{(2)} \times Y_2(\hat{q})]^{(0)}. \tag{3}$$

using the multipole expansion of a plane wave. The spherical Bessel function $j_2(qr)$ is proportional to q^2, as $j_2(qr) \sim (qr)^2/5!!$ in the lowest order of expansion, and $V_T(\mathbf{q})$ becomes

$$V_T(\mathbf{q}) \simeq -\frac{4\pi}{15}\sqrt{\frac{8\pi}{3}}[(\sigma_1 \cdot \sigma_2)^{(2)} \times q^2 Y_2(\hat{q})]^{(0)} \int r^6 f_T(r) dr. \tag{4}$$

In this way, we obtain the tensor interaction in momentum space. This is a further, more detailed, and mathematically rigorous way to show that the arguments in the paper by Dong and Shang are invalid.

References
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