Revision of the brick wall method for calculating the black hole thermodynamic quantities†

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The brick wall method was proposed by t'Hooft[1)] for calculating the contributions of a scalar field to the thermodynamic quantities of black holes, and it has become a common tool for studying thermodynamic properties in the spaces with horizons. We examine the method in Rindler space, which has been often used as a near horizon approximation for black holes, by developing an accurate numerical method for the calculation. We find that the previous works[1–3)] overestimated the partition function and the entropy by almost 2 orders of magnitude. The origin of the discrepancy between our results and theirs is clarified by repeating the calculations of the latter in our framework. We also carry out the analogous studies for the scalar field in de Sitter space and confirm that our method is applicable to the important case of spherically symmetric spaces.

Rindler space is defined by the following coordinate transformation from the Minkowski space.

$$t, x, \mathbf{x}_\perp \to \tau, \xi, \mathbf{x}_\perp : \quad t(\tau,\xi) = \frac{1}{a} e^{a\xi} \sinh a\tau, \quad x(\tau,\xi) = \frac{1}{a} e^{a\xi} \cosh a\tau, \quad (1)$$

where a is the acceleration of a particle at $\xi = \tau = 0$ ($x = \frac{1}{a}, t = 0$) in Minkowski space. In the following calculation, we use $a = 1$, measuring all quantities in units of powers of a. A noninteracting scalar field with mass m satisfies the equation of motion,

$$(\partial_\tau^2 - \partial_\xi^2 - (\nabla_\perp^2 - m^2)e^{2\xi})\phi(\tau,\xi,\mathbf{x}_\perp) = 0. \quad (2)$$

The solution vanishing asymptotically at $\xi \to \infty$ is

$$\phi(\tau,\xi,\mathbf{x}_\perp) = e^{-i\omega\tau} e^{i\mathbf{k}_\perp \mathbf{x}_\perp} K_{i\omega}(\sqrt{m^2 + k_\perp^2} e^\xi), \quad (3)$$

where K is the MacDonald function.

In order to calculate the thermodynamic quantities, we restrict the system in the transverse directions to a square with area $\mathcal{A}$ using the periodic boundary condition in the usual way. The brick wall method uses the Dirichlet boundary condition at ξ_0 with the distance from the horizon $l = e^{\xi_0}$ on the order of the Planck length, l_P. The eigenvalue equation for ω is

$$K_{i\omega}(\sqrt{k_\perp^2 + m^2} e^{\xi_0}) = 0, \quad (4)$$

which results in discrete spectra $\omega_n(k_\perp), n = 1, 2, \ldots$ for each $k_\perp$. The partition function is then given by

$$\ln Z(\beta) = -\frac{\mathcal{A}}{2\pi} \sum_{n=1}^{\infty} \int_0^\infty k_\perp dk_\perp \ln(1 - e^{-\beta\omega_n(k_\perp)}), (5)$$

where $\beta = 2\pi$ at the Unruh temperature.

In the previous works, the eigenvalues of ω are obtained by the WKB approximation and the sum over n for the partition function (5) is replaced by the integral from 0 to ∞ under the assumption that many n's contribute to the sum, leading to the well known closed expressions for the thermodynamic quantities[1–3)] in the case of a massless scalar field. We have solved the equation for $m = 0$ numerically with high accuracy and found that only a small number of n's contribute to the sum with $n = 1$ term dominating at the Unruh temperature. The origin of the overestimation in the previous works is the contribution of the region $0 \le n < 1$, where the spectrum is absent.

The results of our calculation for the partition function $\ln Z$ and the entropy $S = (1 - \beta\partial_\beta) \ln Z$ at the Unruh temperature based on the numerical solutions of eq.(4) are shown in TABLE I together with the closed expressions of the previous works (WKB). They are given in units of $\mathcal{A}/4l^2$ which equals the Beckenstein-Hawking entropy for $l = l_P$. Our result for $\ln Z$ with only the $n = 1$ term in (5) is also shown, demonstrating the dominance of this term. The previous works are seen to overestimate $\ln Z$ by 68 and S by 37.

TABLE I $\ln Z$ and S in units of $\mathcal{A}/4l^2$

	$\ln Z$ $(n = 1)$	$\ln Z$ (total)	S
eq.(4)	1.27×10^{-5}	1.30×10^{-5}	9.68×10^{-5}
WKB		$1/360\pi$	$1/90\pi$

Another important difference between our calculation and the previous ones is that ours is sensitive to the boundary condition at the brick wall because of the sensitivity of eigenvalues for small n's while the previous ones are insensitive owing to the assumption that many n's contribute to the sum in equation (5)

The dominance of the $n = 1$ mode implies that the longitudinal degrees of freedom are practically frozen at the Unruh temperature and the thermodynamic properties are determined by the transverse degrees of freedom.

References
1) G.'t Hooft, Nucl. Phys. **B 256**, 727 (1985).
2) L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994).
3) S. Mukohyama and W. Israel, Phys. Rev. D **58**, 104005 (1998).

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