SSD and the infinite circumference limit of CFT†

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Among all the Virasoro generators of Conformal Field Theory (CFT), three of them, L_0, L_1 and L_{-1}, form a subalgebra that is isomorphic to $sl(2,\mathbb{R})$ and corresponds to the global conformal transformation. The Casimir operator of the subalgebra can be expressed as

$$C_2 = L_0^2 - L_+^2 - L_-^2, \tag{1}$$

where

$$L_+ = \frac{L_1 + L_{-1}}{2}\ ,\ L_- = \frac{L_1 - L_{-1}}{2i}. \tag{2}$$

In analogy with the 2+1 dimensional Lorentz transformation, the space spanned by L_0, L_+ and L_- is apparently divided into three distinctive regions. The first region is the "time-like" region that contains L_0 and small perturbations around it. Any vector within this region can be transformed to L_0 upto some numerical multiplication, by the global conformal transformatiom or the $sl(2,\mathbb{R})$. This is actually the region one would have in mind, when one demanded the invariance of the vacuum on the basis of the physical equivalence for the states connected by the global conformal transformation. The second is the "space-like" region, which contains the linear combination of L_+ and L_-. The region between these two is the last one, and could be called the "light-cone" region. This region is represented by either $L_0 - L_+$ or $L_0 - L_-$.

If one further invokes the analogy with the Lorentz geometry, the "time-like" region corresponds to the "massive" representation. Since one observes the spectrum of L_0 in this region, the "mass" in this case should be the inverse of the circumference, or the finite scale of CFT[1,2]. Then, it is natural to induce that the "light-cone" region corresponds to the "massless" representation and the infinite circumference. In this letter, we will argue that if one takes the generator in the "light-cone" region, say $L_0 - L_+$ (plus anti-holomorphic part $\bar{L}_0 - \bar{L}_+$, to be exact), as a Hamiltonian, one can obtain a CFT with the infinite circumference.

Should we adopt a generator that corresponds to $L_0 - L_+$ as a Hamiltonian, we can define the following conserved charges:

$$\mathcal{L}_\kappa \equiv \frac{1}{2\pi i}\oint^{t=const.} dz\left(-\frac{1}{2}(z-1)^2\right)e^{\frac{2\kappa}{z-1}}T(z), \tag{3}$$

where $T(z) = T_{zz}(z)$ is the energy momentum tensor of the original CFT. Note that for $\kappa = 0$

$$\mathcal{L}_0 = \frac{1}{2\pi i}\oint^{t=const.} dz\left(-\frac{1}{2}(z-1)^2\right)T(z)$$

$$= L_0 - \frac{L_1 + L_{-1}}{2}. \tag{4}$$

One can further calculate the commutation relations among the charges defined above using the operator product expansion of the energy momentum tensor

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots. \tag{5}$$

The result reads

$$[\mathcal{L}_\kappa, \mathcal{L}_{\kappa'}] = (\kappa - \kappa')\mathcal{L}_{\kappa+\kappa'} + \frac{c}{12}\kappa^3\delta(\kappa+\kappa'). \tag{6}$$

We have thus obtained the continuous Virasoro algebra with the central charge c, establishing that we have the theory that exhibits the continuous spectrum. This is consistent with the argument presented at the beginning.

This also nicely explains the feature observed in the phenomena called sine-square deformation (SSD) at least for the case that involves CFT. It was found [3–6] that a certain class of quantum systems, systems with closed and open boundary conditions, have identical vacua provided that the coupling constants of the open-boundary system are modulated in a specific way. In particular, SSD works for two-dimensional conformal field theories and it's implications for string theory were discussed by the present author[7,8]. SSD for CFT adopts exactly (4) as the (holomorphic part of) Hamiltonian. At that time, it had been somewhat enigmatic that these two systems with different boundary conditions share the same vacuum state, but this can be explained through the discovery of the continuous spectrum for the SSD system. Because the continuous spectrum suggest that SSD system has an infinitely large space, the distinction between the open and closed condition at the ends that located at infinitely away, is no longer physically relevant.

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References

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