Linearity of quantum probability measure and Hardy's model†

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Hardy proposed a characterization of entanglement that does not use inequalities by EPR-type arguments. It is however disturbing that his scheme, which is intended as a measure of entanglement, completely fails for the maximally entangled case.[1)]

The local hidden-variables model in $d = 4 = 2 \times 2$ dimensions of the Hilbert space is defined by[2)]

$$\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle = \int_\Lambda \rho(\lambda)d\lambda a(\psi,\lambda)b(\psi,\lambda), \quad (1)$$

where $\mathbf{a}$ and $\mathbf{b}$ are 3-dimensional unit vectors, σ stands for the Pauli matrix, and $a(\psi,\lambda)$ and $b(\psi,\lambda)$ are dichotomic variables assuming the eigenvalues ±1 of $\mathbf{a}\cdot\sigma$ and $\mathbf{b}\cdot\sigma$, respectively. One can show that this local hidden-variables model does not satisfy the linearity of the quantum mechanical probability measure in the sense $\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle + \langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma|\psi\rangle = \langle\psi|\mathbf{a}\cdot\sigma\otimes(\mathbf{b}+\mathbf{b}')\cdot\sigma|\psi\rangle$ for non-collinear $\mathbf{b}$ and $\mathbf{b}'$. If the linearity of the probability measure is strictly imposed, which is tantamount to asking that the non-contextual hidden-variables model in $d = 4$ gives the CHSH inequality $|\langle B\rangle| \le 2$ uniquely,[3)] it is shown that the hidden-variables model can describe only separable quantum mechanical states[4)]

$$\langle\psi|\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma|\psi\rangle = \int \rho_1(\lambda_1)d\lambda_1 a(\psi,\lambda_1) \times \int \rho_2(\lambda_2)d\lambda_2 b(\psi,\lambda_2). \quad (2)$$

In this case, it is shown that Hardy's model becomes trivial. Although Hardy's paradox is interesting as an experimental test of local realism, its mathematical basis is less solid than hitherto assumed.

Hardy's model

Hardy defines the projection operators[1)]

$$\hat{U}_i = |u_i\rangle\langle u_i|, \quad \hat{D}_i = |d_i\rangle\langle d_i|, \quad (3)$$

with $i = 1, 2$, and

$$|u_i\rangle = \frac{1}{\sqrt{\alpha+\beta}}[\beta^{1/2}|+\rangle_i + \alpha^{1/2}|-\rangle_i],$$
$$|d_i\rangle = \frac{1}{\sqrt{\alpha^3+\beta^3}}[\beta^{3/2}|+\rangle_i - \alpha^{3/2}|-\rangle_i] \quad (4)$$

for the entangled state $|\psi\rangle = \alpha|+\rangle_1|+\rangle_2 - \beta|-\rangle_1|-\rangle_2$ with $\alpha^2 + \beta^2 = 1$.

He then shows the relations

$$\frac{\langle\psi|D_1U_2D_1|\psi\rangle}{\langle\psi|D_1|\psi\rangle} = 1, \quad (5)$$
$$\frac{\langle\psi|D_2U_1D_2|\psi\rangle}{\langle\psi|D_2|\psi\rangle} = 1, \quad (6)$$
$$\frac{\langle\psi|D_1D_2D_1|\psi\rangle}{\langle\psi|D_1|\psi\rangle} = 1 - \frac{\alpha\beta}{(1-\alpha\beta)}, \quad (7)$$
$$\langle\psi|U_1U_2|\psi\rangle = 0, \quad (8)$$

with $0 < \alpha\beta \le 1/2$.

In the hidden-variables model, the projection operators are assigned their eigenvalues; for example, $D_1(\psi,\lambda) = 1$ or 0, depending on the hidden-variable λ. Relation (7) implies $\int d\lambda\rho(\lambda)D_1(\psi,\lambda)D_2(\psi,\lambda) \neq 0$ for $0 < \alpha\beta < 1/2$ and thus

$$D_1(\psi,\lambda) = 1 \text{ and } D_2(\psi,\lambda) = 1 \quad (9)$$

for *some* λ, while (8) implies $\int d\lambda\rho(\lambda)U_1(\psi,\lambda)U_2(\psi,\lambda) = 0$ and thus

$$U_1(\psi,\lambda)U_2(\psi,\lambda) = 0 \quad (10)$$

for all λ. On the other hand relations (5) and (6) imply

$$D_1(\psi,\lambda) = 1 \Rightarrow U_2(\psi,\lambda) = 1,$$
$$D_2(\psi,\lambda) = 1 \Rightarrow U_1(\psi,\lambda) = 1, \quad (11)$$

respectively, where $\Rightarrow$ means "inevitably implies".

For the entangled state with $0 < \alpha\beta < 1/2$ except for $\alpha\beta = 1/2$, which implies the maximum entanglement, the relations (9)-(11) are inconsistent.[1)] This is called Hardy's paradox, which shows the inconsistency of local realism with entanglement except for the maximally entangled case without referring to inequality.

On the other hand, for a pure state, Bell's theorem $|\langle B\rangle| \le 2$ with $B = \mathbf{a}\cdot\sigma\otimes(\mathbf{b}+\mathbf{b}')\cdot\sigma + \mathbf{a}'\cdot\sigma\otimes(\mathbf{b}-\mathbf{b}')\cdot\sigma$ for any $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ implies[5)] relation (2), namely, the separable state. The separable state in Hardy's model, which is consistent with local realism, imlies $\alpha = 1$ and $\beta = 0$, for example, for which $|\psi\rangle = |+\rangle_1|+\rangle_2$ while $\hat{U}_i = |-\rangle_{ii}\langle-|$ and $\hat{D}_i = |-\rangle_{ii}\langle-|$. In this case,

$$\langle U_1\rangle = \langle U_2\rangle = \langle D_1\rangle = \langle D_2\rangle = 0, \quad (12)$$

and all the correlations vanish; thus, Hardy's model becomes mathematically trivial. Hardy's model is inconsistent with local realism by construction.

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