Sine square deformation and its implications to string theory†

T. Tada*1

It was recently found that certain 1d quantum systems with an open boundary condition can share the same vacuum state with a similar system having a closed boundary condition, if the coupling constants of the system with an open boundary are modulated in a certain way called *sine square deformation*[1–3]. Sine square deformation works similarly in two-dimensional conformal field theories, which describe string theory [4]. We have investigated sine square deformation in the context of string theory, focusing in particular on open/closed duality.

Sine square deformation is the modulation of the coupling of open boundary systems so that

$$J_{i,i+1} \equiv J \sin^2\left(\frac{n}{N}\pi\right) \tag{1}$$

keeping the boundary coupling $J_{0,1} = J_{N,N+1} = 0$ at the both ends (Fig. 1) for the following 1d quantum system:

$$\mathcal{H} = -\sum J_{n,n+1}\left(\sigma_n \cdot \sigma_{n+1}\right). \tag{2}$$

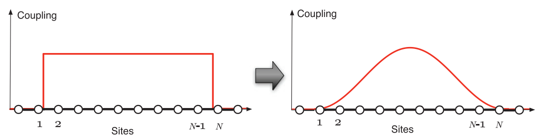

Fig. 1. Sine square deformation of the coupling for a 1d quantum system.

In the case of conformal field theory, the Hamiltonian $\mathcal{H}_{\rm SSD}$ that is sine square deformed is

$$\frac{\pi}{l}\left(L_0 + \bar{L}_0 - \frac{L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}}{2}\right) - \frac{\pi c}{12l}, \tag{3}$$

while the original Hamiltonian is

$$\mathcal{H}_0 = \frac{2\pi}{l}\left(L_0 + \bar{L}_0\right) - \frac{\pi c}{6l}. \tag{4}$$

Then, the vacuum $|0\rangle$ for $\mathcal{H}_0$ is also the vacuum of $\mathcal{H}_{\rm SSD}$ with half the energy.

To interpret the sine square deformation in terms of the dynamics of the world sheet, we need to find the corresponding Lagrangean. We found that the Lagrangean corresponding to $\mathcal{H}_{\rm SSD}$ is obtained by taking α to 1 in the following expression:

$$\mathcal{L}_\alpha = \frac{1}{2}\int dx\left\{\left(\partial_t\varphi\right) f_t\left(\partial_t\varphi\right) - \left(\partial_x\varphi\right) f_x\left(\partial_x\varphi\right)\right\}, \tag{5}$$

where

$$f_x(x) = 1 - \alpha\cos\frac{x}{l}, f_t(x) = N\sum_{k\in Z} r^{|k|} e^{2\pi ikx/l}, \tag{6}$$

and

$$r \equiv \frac{1-\sqrt{1-\alpha^2}}{\alpha},\ N \equiv \frac{1}{\sqrt{1-\alpha^2}}. \tag{7}$$

One can readily see that the g_{00} component of the world sheet metric in the Lagrangean, f_t, diverges severely as we apply sine square deformation. This is in some sense expected because at the SSD point there occurs an event as singular as the change of the boundary condition.

One can apply sl(2, C) transformation $e^{a\frac{L_1 - L_{-1}}{2}}$ to (the holonomic part of) $\mathcal{H}_0$ to obtain

$$\cosh a L_0 - \sinh a\frac{L_1 + L_{-1}}{2}. \tag{8}$$

The right-hand side of the above would have corresponded to $\mathcal{H}_{\rm SSD}$ if $\cosh a = \sinh a$, which is a direct contradiction with the identity $\cosh^2 a - \sinh^2 a = 1$. One, therefore, needs to take $a \to \infty$ and suitably rescale. Hence, $\mathcal{H}_{\rm SSD}$ is not connected with $\mathcal{H}_0$ through the ordinary sl(2,c) transformation, but through a certain limiting procedure.

We also found that $\mathcal{H}_{\rm SSD}$ has the following different vacua other than $|0\rangle$

$$e^{L_{-1}}|h\rangle, \tag{9}$$

where $|h\rangle$ is the state corresponding to the primary fields of CFT. However, the norm of (9) is divergent. One also needs a certain limiting process to properly define (9).

In summary, we have investigated sine square deformation of string theory to shed light on the relation between open and closed strings. Recent studies of string dualities suggest that one needs to go beyond the realm where open and closed are inseparable to understand the true dynamics of string theory. We hope that we can further uncover the nature of this realm through insights offered by sine square deformation.

References

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† Condensed from the article in Proceedings of the 12th Asia Pacific Physics Conference (APPC12)

*1 RIKEN Nishina Center