Conditionally valid uncertainty relations†

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It is shown that the well-defined unbiased measurement or disturbance of a dynamical variable is not maintained for the precise measurement of the conjugate variable, independently of uncertainty relations. The conditionally valid uncertainty relations on the basis of those additional assumptions, which include most of the familiar Heisenberg-type relations, thus become singular for the precise measurement. We clarify some contradicting conclusions in the literature concerning those conditionally valid uncertainty relations: The failure of a naive Heisenberg-type error-disturbance relation and the modified Arthurs-Kelly relation in the recent spin measurement is attributed to this singular behavior. The naive Heisenberg-type error-disturbance relation is formally preserved in quantum estimation theory, which is shown to be based on the strict unbiased measurement and disturbance, but it leads to unbounded disturbance for bounded operators such as spin variables. In contrast, the Heisenberg-type error-error uncertainty relation and the Arthurs-Kelly relation, as conditionally valid uncertainty relations, are expected to be consistently maintained.

A recent experiment[1)], which invalidated a naive Heisenberg-type error-disturbance relation[2)], revived our interest in the subject of uncertainty relations. In contrast to the naive Heisenberg-type error-disturbance relation, the relations which are based on only the positive definite Hilbert space and natural commutator algebra are expected to be valid as long as quantum mechanics is valid, namely, "universally valid"[2)3)]. It was recently shown[4)] that all the known universally valid uncertainty relations are derived from Robertson's relation written for suitable combinations of operators. It is important to distinguish the uncertainty relations which are universally valid from those relations based on additional assumptions and thus only conditionally valid.

In this paper, we analyze the implications of the assumptions of unbiased joint measurements or unbiased measurement and disturbance which are widely used in the formulation of uncertainty relations[5)]. We clarify the origin of quite different conclusions concerning the conditionally valid Heisenberg-type relations in the measurement operator formalism[2)] and in the quantum estimation theory[6)] which is a new approach to uncertainty relations.

We first note that the well-defined unbiased measurement or disturbance of a quantum mechanical operator is not maintained for the precise measurement of the conjugate operator in the framework of the ordinary measurement theory. For example, those assumptions lead to

$$\langle [M^{out}, N^{out}] \rangle = \langle [A, B] \rangle,$$
$$\langle [M^{out}, B^{out}] \rangle = \langle [A, B] \rangle. \tag{1}$$

We work in the Heisenberg picture and the variables without any suffix stand for the initial variables; A, B stand for dynamical variables and M, N stand for the corresponding measurement operators, respectively. The variables $M^{out} = U^\dagger(1 \otimes M)U$ and $N^{out} = U^\dagger(1 \otimes N)U$ stand for the variables after the measurement, and $B^{out} = U^\dagger(B \otimes 1)U$ stands for the variable B after the measurement of A. By assumption, $\langle [M^{out}, N^{out}] \rangle = \langle [M^{out}, B^{out}] \rangle = 0$, and thus relations in (1) are contradictions.

The conditionally valid uncertainty relation such as naive Heisenberg-type error-disturbance relation[1)2)],

$$\sigma(M^{out} - A)\sigma(B^{out} - B) \geq \frac{1}{2}|\langle [A, B] \rangle|, \tag{2}$$

which is based on the assumptions of unbiased measurement and disturbance, thus fails if one formulates the relation in terms of well-defined bounded operators. The naive Heisenberg-type error-disturbance relation is formally preserved in quantum estimation theory, but the disturbance of the bounded operator is forced to be singular and divergent for the precise measurement of the conjugate variable[6)].

In contrast, the Heisenberg-type error-error uncertainty relation

$$\sigma(M^{out} - A)\sigma(N^{out} - B) \geq \frac{1}{2}|\langle [A, B] \rangle|, \tag{3}$$

and the Arthurs-Kelly relation,

$$\sigma(M^{out})\sigma(N^{out}) \geq |\langle [A, B] \rangle|, \tag{4}$$

as conditionally valid uncertainty relations, are expected to be consistently maintained.

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