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# General conditions ensuring relativistic causality in an effective field theory based on the derivative expansion†

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We discuss the general conditions ensuring relativistic causality in an effective field theory based on the derivative expansion. Relativistic causality implies that the Green function vanishes in a space-like region. It is known that a naive derivative expansion violates causality in some cases such as the first-order relativistic dissipative hydrodynamics. We note that the Lorentz covariance and time and space derivatives of equal order do not ensure causality. We derive the general conditions for causality that should be satisfied by any effective theory consistent with special relativity.

Derivative expansion is a useful tool at a low-energy scale and widely used in effective theories. Chiral perturbation theory is a good example of a successful low-energy effective theory in hadron physics[1]. From a modern perspective, hydrodynamics is also a low-enegy effective theory; the leading-order hydrodynamic equations are called Euler equations, and the first-order hydrodynamic equations are called Navier–Stokes equations.

Causality is an important concept in physics. In relativistic systems, the propagation of any information cannot exceed the speed of light (relativistic causality). However, it seems that a low-energy effective theory in medium is incompatible with relativistic causality. For example, in first-order relativistic hydrodynamics, shear and heat flows violate causality because the first-order equation has the form of a diffusion equation[2,3]. We note that the first-order hydrodynamic equation is Lorentz covariant, i.e., the covariance does not ensure the causality. It is argued that the acausality of the first-order hydrodynamics originates from the difference between the order of time- and space-like derivatives in the equation of motion. In the diffusion equation, the time-like derivative is of the first order, while the space-like one is of the second order. However, we note that equality in the order of time- and space-like derivatives does not ensure the causality. For example, let us consider the following equation:

$$\left[\tau(u^\mu\partial_\mu)^2+u^\mu\partial_\mu+\Gamma(\eta^{\mu\nu}-u^\mu u^\nu)\partial_\mu\partial_\nu\right]n(x^\mu)=0,\quad(1)$$

where $n(x)$ is a scalar density, $u^\mu$ is a constant time-like vector, $\eta=\mathrm{diag}(-1,1,1,1)$,$\tau$ is the relaxation time, and $\Gamma$ is the diffusion constant. In this equation, the time-like derivative is of the same order as the space-like one. This equation has the form of the telegraphic equation such that the propagation is restricted in the region defined by $vx_t > x_s$, where $v=\sqrt{\Gamma/\tau}$, $x_t=u\cdot x$, and $x_s=\sqrt{-(\eta^{\mu\nu}-u^\mu u^\nu)x_\mu x_\nu}$. If $\Gamma<\tau$, causality is satisfied because the velocity is smaller than the speed of light, i.e., $v<1$[3]. However, if $\Gamma>\tau$, the propagation speed exceeds the speed of light. Furthermore, if $\Gamma=0$, causality is not violated even though the order of the time and space derivatives are different. Therefore, the equal order of space and time derivatives in the equation of motion does not ensure in itself that the Green function is causal.

What ensures the causality in general?

The purpose of this paper is to derive the conditions ensuring relativistic causality in an effective theory based on the derivative expansion. We will consider the retarded Green function in a scalar theory at tree level, i.e., thermal and quantum fluctuations will not be taken into account. In this case, the retarded Green function in the derivative expansion is generally written as a rational function in the momentum space:

$$G_R(\omega,k)=\frac{Q(\omega,k)}{P(\omega,k)},\quad(2)$$

where $P(\omega,k)$ and $Q(\omega,k)$ are polynomials in $\omega$ and $k$:

$$P(\omega,k)=p_n(k)\omega^n+p_{n-1}(k)\omega^{n-1}+\ldots+p_0(k),\quad(3)$$
$$Q(\omega,k)=q_m(k)\omega^m+q_{m-1}(k)\omega^{n-1}+\ldots+q_0(k).\quad(4)$$

Here, $n>m$, and $p_j(k)$ and $q_j(k)$ are the polynomials in $k$. Because we assumed isotropy, the Green function turns becomes a function of $k\equiv|\boldsymbol{k}|$. The relativistic causality implies that the retarded Green function must vanish in the space-like region. Therefore, we derive the general condition ensuring Eq. (2) vanishes in the space-region, which is given by

$$\lim_{k\to\infty}\left|\mathrm{Re}\frac{\omega(k)}{k}\right|<1 \text{ and } \lim_{k\to\infty}\left|\mathrm{Im}\frac{\omega(k)}{k}\right|<\infty,\quad(5)$$

and the condition that $p_n(k)$ must not depend on $k$. Here, $\omega(k)$ is a pole of Eq. (2). These conditions ensure causality in effective theories based on the derivative expansion, and they are the main results of our paper.

References
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